Journal of Statistical Physics, Vol. 35, Nos. 1/2, 1984

The Mean Spherical Model in a Random External Field and the Replica Method

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Received July 28, 1983

We provide a quick elementary solution of the mean spherical model in a random external field. This also allows an immediate proof of the self-averaging property of the free energy. We calculate the free energy by means of the replica method, i.e., for any (not necessarily integer) "replica number" n, and show that when a phase transition occurs the limits ($\lim_{H \to 0_+} \lim_{N \to \infty}$) and $n \to 0$ are not interchangeable.

KEY WORDS: Mean spherical mode; random external field; self-averaging property; critical dimension; replica method; thermodynamic convergence.

1. INTRODUCTION

The spherical model in a random external field(1) is one of the few exactly soluble models which exhibit $\nu = 4$ as the lower critical dimension for the existence of random ferromagnetism. A simple, "one-sentence" solution of the model seems therefore desirable. The very general approach of Ref. 1 is, however, somewhat abstract. In this note, we provide a quick elementary solution of the model, which also allows an immediate proof of the self-averaging(2,3) property. The latter is essential to account for the reproducibility of the outcomes of experiments realized on random systems. We also show that the free energy may be computed exactly[3] by the replica method (Refs. 2 and 4 and references given there) even if the "number of replicas" is not an integer, thereby illustrating some of the general results of

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0022-4715/84/0400-0089$03.50/0

Refs. 2 and 4. In particular, if $\nu \geqslant 5$, and the variance of the external field is sufficiently small while the inverse temperature β is sufficiently large, there is a phase transition [1] and the convergence is not thermodynamic [2] with respect to the double limit $\lim_{H\to 0}\lim_{N\to\infty}$ (where H is the external field, and "$N \to \infty$" denotes the infinite-volume limit). We say that $N^{-1}W_N$ converges *thermodynamically* to α, if for any $\delta > 0$ we can find a constant $c = c(\delta) > 0$ such that for all sufficiently large N,

$$\text{Prob}\left\{ |N^{-1}W_N - \alpha| \geqslant \delta \right\} \leqslant \exp(-cN)$$

Accordingly, it might be expected that the limits ($\lim_{H\to 0}\lim_{N\to\infty}$) and $n \to 0$ in the replica method are not interchangeable for some quantities. Indeed, this is explicitly verified.

2. THE MEAN SPHERICAL MODEL

Let Λ_N denote a hypercube $\{-L, \ldots, L\}^\nu$ enclosing $N = (2L+1)^\nu$ points in Z^ν. The Hamiltonian restricted to Λ_N of the mean spherical model in a random external field may be written

$$H_N(\boldsymbol{\phi}) \equiv (\boldsymbol{\phi}, -\Delta/2\boldsymbol{\phi}) - \mu(\boldsymbol{\phi}, \boldsymbol{\phi}) - H \sum_{x \in \Lambda_N} \phi(x) - (\mathbf{h}, \boldsymbol{\phi}) \tag{1}$$

We denote by boldface characters a general N-component vector whose components are labeled by the points of Λ_N. $\boldsymbol{\phi}$ is a classical "spin vector" whose components range over $\mathbb{R}$, μ is the chemical potential, H is the (nonrandom) external field, and $(-\Delta)$ the "lattice Laplacean"

$$(-\Delta\phi)(x) = 2\nu\phi(x) - \sum_{i=1}^{\nu} \left[\phi(x+e_i) + \phi(x-e_i) \right]$$

where e_i, $i = 1, \ldots, \nu$ is a unit vector in the ith direction. The scalar product between two vectors $\mathbf{y}$ and $\mathbf{z}$ is

$$(\mathbf{y}, \mathbf{z}) = \sum_{x \in \Lambda_N} \bar{y}(x) z(x)$$

We assume periodic boundary conditions in (1). The vector $\mathbf{h}$ is a random vector representing the random external field, whose components $h(x)$, $x \in Z^\nu$, are assumed to be independent identically distributed random variables, with probability distribution ρ, mean zero, and covariance σ:

$$\langle h(x) \rangle_\rho = 0, \qquad \forall x \in \mathbb{Z}^\nu \tag{2}$$

$$\langle h(x) h(y) \rangle_\rho = \sigma^2 \delta_{x,y}, \qquad x, y \in \mathbb{Z}^\nu \tag{3}$$

The partition function is defined by

$$Z_N(\beta, \mu, H) = \int d\boldsymbol{\phi} \exp\left[-\beta H_N(\boldsymbol{\phi}) \right]$$

where $d\phi \equiv \prod_{x \in \Lambda_N} d\phi(x)$, and the free energy by

$$f_N(\beta, \mu, H) = -\frac{\beta^{-1}}{N} \log Z_N(\beta, \mu, H) \tag{4}$$

The Gibbs expectation value of an observable $F(\phi)$ is

$$\langle F \rangle_N \equiv \frac{\int d\phi \exp[-\beta H_N(\phi)] F(\phi)}{Z_N(\beta, \mu, H)}$$

We require ϕ to satisfy the spherical constraint

$$\langle (\phi, \phi) \rangle_N = N \tag{5a}$$

or, in terms of μ:

$$\left(\frac{\partial f_N(\beta, \mu, H)}{\partial \mu} \right)_{\beta, H} = -1 \tag{5b}$$

Finally, the Fourier transform $\hat{f}$ of a function f is defined by

$$\hat{f}(k) = \frac{1}{\sqrt{N}} \sum_{x \in \Lambda_N} e^{-ikx} f(x) \qquad \forall k \in \Lambda_N^* = \left\{ \frac{\pi x}{L+1}, x \in \Lambda_N \right\}.$$

Let A be a *strictly positive* matrix. We have

$$\int d\mathbf{x} \exp\left[-\tfrac{1}{2}(\mathbf{x}, A\mathbf{x}) + (\mathbf{y}, \mathbf{x})\right] = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} \exp\left[\tfrac{1}{2}(\mathbf{y}, A^{-1}\mathbf{y})\right] \tag{6}$$

By (1) we see that $Z_N(\beta, \mu, H)$ is of the form of the left-hand side of (6), with the following identifications:

$$A = \beta(-\Delta - 2\mu)$$

$$\mathbf{y} = \beta H \mathbb{1} + \mathbf{h}, \qquad \mathbb{1} \equiv \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

This yields immediately

$$f_N(\beta, \mu, H) = -\frac{\beta^{-1}}{2} \log(2\pi) + \frac{\beta^{-1}}{2N} \operatorname{tr} \log A + \frac{H^2}{4\mu} + \frac{\beta H}{2\mu N} (\mathbb{1}, \mathbf{h}) - \frac{\beta}{2N} (\mathbf{h}, A^{-1}\mathbf{h}) \tag{7}$$

3. THE FREE ENERGY

To compute the averaged free energy we only need

$$\langle(\mathbf{h}, A^{-1}\mathbf{h})\rangle_\rho = \sum_{x,y} (A^{-1})_{xy} \cdot \langle h(x)h(y)\rangle_\rho$$

$$= \sigma^2 \operatorname{tr} A^{-1} = \frac{\sigma^2}{2\beta} \sum_{k \in \Lambda_N^*} \frac{1}{\epsilon(k) - \mu} \tag{8}$$

where

$$\epsilon(k) \equiv \sum_{i=1}^{\nu} (1 - \cos k_i), \qquad k \equiv (k_i)_{i=1}^{\nu}$$

The following results follow almost immediately from (7) and (8) (see Ref. 5 or 6). If $H \neq 0$, the equation

$$\left(\frac{\partial \langle f_N(\beta, \mu, H)\rangle_\rho}{\partial \mu} \right)_{\beta, H} = -1$$

has a unique solution $\mu_N(\beta, H)$ such that

$$\mu_N(\beta, H) \xrightarrow[N\to\infty]{} \mu(\beta, H)$$

If $H \neq 0$,

$$\mu(\beta, H) < 0$$

is the *unique* solution of the equation

$$\frac{H^2}{4\mu^2} + \frac{\beta^{-1}}{2(2\pi)^\nu} \int_B d^\nu k \frac{1}{\epsilon(k) - \mu} + \frac{\sigma^2}{4(2\pi)^\nu} \int_B d^\nu k \frac{1}{(\epsilon(k) - \mu)^2} = 1 \tag{9}$$

where $B \equiv [-\pi, \pi]^\nu$ is the first Brillouin zone. Furthermore,

$$\lim_{N\to\infty} \langle f_N(\beta, \mu = \mu_N(\beta, H), H)\rangle_\rho = \langle f(\beta, \mu(\beta, H), H)\rangle_\rho \tag{10}$$

where

$$\langle f(\beta, \mu, H)\rangle_\rho = \frac{H^2}{4\mu} - \frac{\beta^{-1}}{2(2\pi)^\nu} \int_B d^\nu k \left\{ \log(2\pi) - \log[2\beta(\epsilon(k) - \mu)] \right\}$$

$$- \frac{\sigma^2}{4(2\pi)^\nu} \int_B d^\nu k \frac{1}{\epsilon(k) - \mu} \tag{11}$$

The quantity

$$\langle m_N(\beta, H)\rangle_\rho = -\left[\left(\frac{\partial\langle f_N(\beta, \mu, H)\rangle_\rho}{\partial H}\right)_{\beta,\mu}\right]_{\mu=\mu_N(\beta,H)}$$
$$= -\frac{H}{2\mu_N(\beta, H)} \tag{12}$$

is the magnetization. In this way we arrive at our first proposition.

Proposition 1. If $\nu \leqslant 4$

$$\lim_{H\to 0_+} \mu(\beta, H) = \mu(\beta, 0) < 0 \tag{13}$$

and

$$\lim_{H\to 0_+} \lim_{N\to\infty} \langle m_N(\beta, H)\rangle_\rho = 0 \tag{14}$$

If $\nu \geqslant 5$ and

$$\frac{\sigma^2}{4(2\pi)^\nu}\int_B d^\nu k \frac{1}{\epsilon(k)^2} < 1$$

then, for β sufficiently large,

$$\lim_{H\to 0_+} \mu(\beta, H) = 0 \tag{15}$$

and

$$\lim_{H\to 0_+} \lim_{N\to\infty} \langle m_N(\beta, H)\rangle_\rho = \gamma_\beta \neq 0 \tag{16}$$

We now consider $f_N(\beta, \mu, H)$ given by (4) (*no* averaging). We have[4]

$$(\mathbf{h}, A^{-1}\mathbf{h}) = \frac{1}{\beta}(\mathbf{h}, G\mathbf{h})$$

where

$$G = (-\Delta - 2\mu)^{-1}$$

Hence, if $\mu < 0$,

$$|G(x, y)| \leqslant c\frac{\exp(2\mu\|x-y\|)}{\|x-y\|^{(\nu-1)/2}} \tag{17}$$

[4] In the following argument we ignore the fact that A, hence G, depends on N, but it is easy to verify that the restriction of G to Λ_N (with periodic boundary conditions) also obeys (17) (with c independent of N).

for $\|x-y\|$ large ($\|x\|=(x_1^2+\cdots+x_\nu^2)^{1/2}$ if $x\equiv(x_i)_{i=1}^\nu$). By (7) and (17), $f_N(\beta,\mu,H)$ converges with probability 1 to $\langle f(\beta,\mu,H)\rangle_\rho$ and the convergence is thermodynamic[2] if $\mu<0$. The reason is that we may split up a large "block" $(\mathbf{h}, A^{-1}\mathbf{h})$ into finitely many independent sub-blocks plus an error term which becomes negligibly small as the size of the sub-blocks also goes to infinity. In addition

$$\frac{\partial f_N(\beta,\mu,H)}{\partial\mu}=-\frac{\beta^{-1}}{2N}\sum_{k\in\Lambda_N^*}\frac{1}{\epsilon(k)-\mu}-\frac{H^2}{4\mu^2}$$
$$-\frac{1}{2N}(\mathbf{h},(G*G)_N\mathbf{h})-\frac{\beta H}{2\mu^2N}(\mathbb{1},\mathbf{h}) \tag{18}$$

where

$$(G*G)_N(x-z)\equiv\sum_{y\in\Lambda_N}G(x-y)G(y-z) \tag{19}$$

Clearly, if $\mu<0$,

$$|(G*G)_N(x)|\leqslant c\exp(-\alpha\|x\|) \tag{20}$$

with $\alpha>0$ and c independent of N. It follows easily (see, e.g., Ref. 2) from (18) and (20) that

$$\frac{\partial f_N(\beta,\mu,H)}{\partial\mu}\xrightarrow[N\to\infty]{}\frac{\partial\langle f(\beta,\mu,H)\rangle_\rho}{\partial\mu}\qquad\text{with probability 1} \tag{21}$$

if $\mu<0$. For $H\neq0$ (5b) has a unique solution $\tilde{\mu}_N(\beta,H)<0$. A simple argument using (18)–(21) shows that, if $H\neq0$,

$$\tilde{\mu}_N(\beta,H)\xrightarrow[N\to\infty]{}\mu(\beta,H)<0\qquad\text{with probability 1} \tag{22}$$

where $\mu(\beta,H)$ is the unique solution of (9). Equation (22) has the following important consequence:

Proposition 2. The following statements hold with probability 1. If $\nu\leqslant4$,

$$\lim_{H\to0_+}\lim_{N\to\infty}\tilde{\mu}_N(\beta,H)=\mu(\beta,0)<0$$

and

$$\lim_{H\to0_+}\lim_{N\to\infty}m_N(\beta,H)=0$$

If $\nu\geqslant5$, under the same conditions stated in Proposition 1,

$$\lim_{H\to0_+}\lim_{N\to\infty}\tilde{\mu}_n(\beta,H)=0 \tag{23}$$

and

$$\lim_{H\to 0_+}\lim_{N\to\infty} m_N(\beta, H) = \gamma_\beta \neq 0 \quad \blacksquare$$

4. THE REPLICA METHOD

We now compare the free energy $\langle f(\beta, \mu, H)\rangle_\rho$ with the one obtained by the replica method. We have to impose the condition ($\mu < 0$)

$$\mu + \frac{\beta n \sigma^2}{2} < 0 \tag{24}$$

and that the probability distribution be *Gaussian*. Define[4]

$$\begin{aligned}\phi_N(n) &= -\frac{\beta^{-1}}{N}\log\langle Z_N^n(\beta, \mu, H)\rangle_\rho \\ &= -\frac{\beta^{-1}}{N}\log\langle \exp[n\log Z_N(\beta, \mu, H)]\rangle_\rho\end{aligned} \tag{25}$$

For the integers n which satisfy (24) we get

$$\begin{aligned}\langle Z_N^n(\beta, \mu, H)\rangle_\rho &= \int d\phi_1\cdots d\phi_n \exp\{-\beta[H_n(\phi_1) + \cdots + H_N(\phi_n)]\} \\ &= \int d\phi_1\cdots d\phi_n \prod_{i=1}^{n}\exp\Big\{-\beta\Big[(\phi_i, -\Delta/2\phi_i) - \mu(\phi_i, \phi_i) \\ &\qquad - H\sum_{x\in\Lambda_N}\phi_i(x)\Big]\Big\} \\ &\qquad \times \langle\exp[\beta(\mathbf{h}, \phi_1 + \cdots + \phi_n)]\rangle_\rho\end{aligned} \tag{26a}$$

with

$$\begin{aligned}&\langle\exp[\beta(\mathbf{h}, \phi_1 + \cdots + \phi_n)]\rangle_\rho \\ &\quad = \exp[\beta^2(\phi_1 + \cdots + \phi_n, \phi_1 + \cdots + \phi_n)\sigma^2/2]\end{aligned} \tag{26b}$$

by the assumption that ρ is Gaussian. The matrix involved in (26b) is

$$B = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \cdots & 1 \end{pmatrix} = n|\mathbb{1}\rangle_n\langle_n\mathbb{1}|$$

where

$$|1\rangle_n \equiv \frac{1}{\sqrt{n}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Hence there exists an orthogonal matrix $0 \equiv (0_{ij})$ such that

$$0^{-1}B0 = \begin{bmatrix} n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Define $\phi' = 0^{-1}\phi$. Then

$$\sum_i \phi_i = \sum_j \phi_j' \Big(\sum_j 0_{ij} \Big), \qquad \text{but}$$

$$\sum_i 0_{ij} = \begin{cases} \sqrt{n}\,, & \text{if } j = 1 \\ 0, & \text{otherwise} \end{cases}$$

We may therefore compute the Gaussian integrals in (26). Only the integral involving ϕ_1 contains H, and it may be computed by the substitution $H \to H\sqrt{n}$ together with

$$\mu \to \mu_n \equiv \mu + \beta n \sigma^2/2 \tag{27}$$

in the free energy of the (nonrandom) spherical model. Note, however, that the integrals (26a) diverge unless $\mu_n < 0$, which is (24). We now find

$$\begin{aligned} \phi_N(n) = &- \frac{\beta^{-1}}{N} \sum_{k \in \Lambda_N^*} \{ \log(2\pi) - \log[2\beta(\epsilon(k) - \mu_n)] \} \\ &- (n-1) \frac{\beta^{-1}}{2N} \sum_{k \in \Lambda_N^*} \{ \log(2\pi) - \log[2\beta(\epsilon(k) - \mu)] \} \\ &+ \frac{(H\sqrt{n})^2}{4\mu_n} \end{aligned} \tag{28}$$

From (28) and (11) we obtain, if $H \neq 0$,

$$\lim_{N\to\infty} \frac{d\phi_N(n)}{dn}\bigg|_{n=0} = \frac{d}{dn} \lim_{N\to\infty} \phi_N(n)\bigg|_{n=0} = \langle f(\beta, \mu, H) \rangle_\rho \tag{29}$$

Equation (29) illustrates the interchangeability of the limits $N \to \infty$ and $n \to 0$ in the case of thermodynamic convergence.[2]

We see from (28) that the replica trick provides the correct answer in all cases, not only those satisfying (24). By this we mean that there are values of the parameters such that (24) is violated *for any integer n* and the above calculation is meaningless.

The above "mystery" is solved by computing $\phi_N(n)$ exactly for *all* real n which satisfy (24). In fact, the condition (24) arises naturally. We need only (6). By a calculation similar to the one which led to (7), we find

$$\exp[n \log Z_N(\beta, \mu, H)] = \exp\left(-\frac{nH^2N}{4\mu}\right) \times \exp\left\{\frac{n\beta^2}{2}\left[(\mathbf{h}, A^{-1}\mathbf{h}) - \frac{H}{\beta\mu}(\mathbb{1}, \mathbf{h})\right]\right\}$$

with the same notation as in (7). Hence

$$\langle \exp[n \log Z_N(\beta, \mu, H)]\rangle_\rho = \exp\left(-\frac{nH^2N}{4\mu}\right)\left(\frac{1}{(2\pi\sigma^2)^{1/2}}\right)^N \times \int d\mathbf{h} \exp\left\{\frac{n\beta^2}{2}\left[(\mathbf{h}, A^{-1}\mathbf{h}) - \frac{H}{\beta\mu}(\mathbb{1}, \mathbf{h})\right]\right\} \times \exp\left[-\frac{1}{2\sigma^2}(\mathbf{h}, \mathbf{h})\right]$$

Using (6) once again,

$$\langle \exp[n \log Z_N(\beta, \mu, H)]\rangle_\rho = \exp\left(\frac{-nH^2N}{4\mu}\right) \times \left(\frac{1}{(2\pi\sigma^2)^{1/2}}\right)^N \frac{(2\pi)^{N/2}}{(\det A')^{1/2}} \times \exp\left\{\frac{1}{2}\left[\frac{-nH\beta}{2\mu}\mathbb{1}, (A')^{-1}\left(-\frac{nH\beta}{2\mu}\mathbb{1}\right)\right]\right\} \tag{30}$$

where

$$A' \equiv \frac{1}{\sigma^2}\mathbb{1}_d - n\beta^2 A^{-1} \qquad \mathbb{1}_d = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

provided A' is strictly positive, i.e., (24) is satisfied. We obtain (28) from (30) by a straightforward calculation.

We see from (28) that Φ_N and its thermodynamic limit Φ are analytic in n for $\operatorname{Re} n < 0$, even if $\mu = 0$. This confirms and substantiates an earlier observation made by van Hemmen and Palmer (unpublished), and rediscovered by Eisele.[7] In the spherical model the limit $n \to 0_-$ (from the left) is therefore more natural. Equations (17) and (23) (compare) with Ref. 2) indicate that the convergence is not thermodynamic with respect to the double limit ($\lim_{H\to 0_+} \lim_{N\to\infty}$). From Proposition 2 we expect that the limits $n \to 0_-$ and ($\lim_{H\to 0_+} \lim_{N\to\infty}$) may not be interchanged (at least for some quantities) if $\nu \geqslant 5$ and the conditions stated in Proposition 1 are satisfied, implying that a phase transition takes place and (23) holds. In fact, consider the magnetization derived from (28):

$$m_{n,N}(\beta, H) \equiv -\frac{\partial \phi_N(n)}{\partial H} = -\frac{nH}{2\mu_n}$$

$$= -\frac{nH}{2\left[\mu_N(\beta, H) + \beta n \sigma^2/2\right]} \tag{31}$$

Let d^- denote left derivative. By (31)

$$\frac{d^-}{dn}\left(\lim_{H\to 0_+} \lim_{N\to\infty} m_{n,N}(\beta, H)\right)_{n=0} = 0$$

but, by (15) and (16)

$$\lim_{H\to 0_+} \lim_{N\to\infty} \left(\frac{d^-}{dn} m_{n,N}(\beta, H)\right)_{n=0} = \gamma_\beta \neq 0$$

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